

The Ramond-Ramond self-dual Five-form's Partition Function on T^{10}

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ABSTRACT

In view of the recent interest in formulating a quantum theory of Ramond-Ramond p -forms, we exhibit an $SL(10, \mathcal{Z})$ invariant partition function for the chiral four-form of Type IIB string theory on the ten-torus. We follow the strategy used to derive a modular invariant partition function for the chiral two-form of the M-theory fivebrane. We also generalize the calculation to self-dual quantum fields in spacetime dimension $2p = 2 + 4k$, and display the $SL(2p, \mathcal{Z})$ automorphic forms for odd $p > 1$. We relate our explicit calculation to a computation of the B-cycle periods, which are discussed in the work of Witten.

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1. Introduction

The calculation reported in this paper extends the methods in [1]. The result is of interest as it involves a Ramond-Ramond chiral form (while in [1] we dealt with a chiral-two form in six dimensions which is a NS form). Although restricted to the ten-torus, the result is useful in view of the analysis in [2, 3, 4] which sets out a method for calculating the partition functions of self-dual Ramond-Ramond fields in Type II string theory on general manifolds and emphasizes their central role in formulating a quantum theory of non-Lagrangian fields.

Our calculation also provides an explicit formula for the partition function of the self-dual p -form G_p on T^{2p} when the dimension of spacetime $2p = 2 + 4k$ is twice odd, (for odd $p > 1$). These dimensions occur since the Hodge star operator $*$ can be used to define a self-dual $p = (1 + 2k)$ -form for Minkowski signature metrics[‡] (which are needed for quantum mechanics) and has $** = 1$. The dimensions $(2 + 4k)$ are the same as those for which Kervaire invariants are defined. The question of modular invariance of the partition function of self-dual quantum fields on general $(2 + 4k)$ -dimensional spacetimes can be related to these invariants through the work of [5, 6, 7].

For the torus compactifications, we identify the actual value of the determinant Δ describing the contribution of the non-zero modes, where the partition function is $Z = \frac{\Theta}{\Delta}$. From the sum over zero modes, we find the theta function Θ and an explicit representation of the period matrix, which alternatively could be derived by a direct assignment of the B-cycle periods [2]. The ten torus example corresponds to zero shift in the Dirac quantization condition for the five-form field strength G_5 , so it can be formulated without applying the elegant K-theory techniques of [2, 3]. Thus our examples do not probe certain fundamental properties of self-dual RR

[‡] If we define the Hodge dual operation which converts a d -form into a $(D-d)$ -form as $(*J)_{M_1 \dots M_{D-d}} \equiv \frac{1}{d!} \epsilon_{M_1 \dots M_{D-d} \mu_1 \dots \mu_d} J^{\mu_1 \dots \mu_d} \frac{1}{\sqrt{|g_D|}}$ where $\epsilon_{M_1 \dots M_{D-d} \mu_1 \dots \mu_d} \equiv g_{M_1 N_1} \dots g_{M_{D-d} N_{D-d}} g_{\mu_1 \nu_1} g_{\mu_d \nu_d} \epsilon^{N_1 \dots N_{D-d} \nu_1 \dots \nu_d}$, then $(**J)^{\mu_1 \dots \mu_d} = (-1)^{(D-d)d} g_D \frac{1}{|g_D|} J^{\mu_1 \dots \mu_d}$. So for Minkowski signature metrics, which have $|g_D| = -g_D$, we have $D = 2d$ with odd d for $** = 1$.

fields on arbitrary manifolds, but nonetheless provide an explicit case of B-lattice basis vectors, which are somewhat involved to calculate directly even on the flat torus with a general metric. Given the complicated nature of actual constructions on general manifolds, we display the $SL(2+4k, \mathcal{Z})$ automorphic forms for quantum self-dual fields on the torus, although the relevance to string theory for $k > 2$ is not apparent.

The first step is, of course, to define the partition function. Given the notorious difficulties in writing down covariant Lagrangians for a chiral form, we will not start with a Lagrangian (which we do not have) and write down the path integral. Rather, as usual in string calculations [8], we compute the trace $\text{tr} e^{-t\mathcal{H}}$ on the twisted ten-torus. To that purpose we define the Hamiltonian and the momenta for our theory along the lines of [1]. Our formalism for the Ramond-Ramond self-dual five-form partition function will be manifestly $SL(9, \mathcal{Z})$ invariant, which combined with an extra $SL(2, \mathcal{Z})$ symmetry we show, will ensure its $SL(10, \mathcal{Z})$ modular invariance on the ten-torus.

While the partition function for the two-dimensional chiral boson in two dimensions is not modular invariant, in [1] we proved that the chiral two-form of the M-theory fivebrane is $SL(6, \mathcal{Z})$ modular invariant on the six-torus. That proof is generalizable to any chiral $2k$ -form, for integer $k > 0$, which corresponds to a $(2k+1)$ -form self-dual field strength in $2+4k$ dimensions. A crucial step is to view T^{2+4k} as the product $T^2 \times T^{4k}$. The extra $SL(2, \mathcal{Z})$ symmetry mentioned above is the one associated with the two-torus T^2 .

As usual, the calculation of the trace factorizes as $Z = Z_{\text{osc}} \cdot Z_{\text{zero modes}}$, where Z_{osc} is the contribution from the sum over the oscillators and $Z_{\text{zero modes}}$ comes from the sum over the zero modes. The sum over the zero modes will have an anomaly under $SL(2, \mathcal{Z})$ which cancels against a similar anomaly in the sum over oscillators.

The sum over the oscillators associated with the three degrees of freedom of the two-form potential of the M-theory fivebrane factored into sums over “massless”

and “massive” modes, according to whether they had zero momentum or non-zero momentum on T^4 . The $SL(2, \mathcal{Z})$ anomaly needed to cancel the anomaly coming from the zero modes was due to the contribution of the ‘2d-massless’ modes, *i.e.* the modes with zero momentum in the transverse directions on T^4 . The modes with non-zero momentum on T^4 behaved like massive scalars in two dimensions, and did not have any anomaly. The overall effect was that the three degrees of freedom of the chiral two-form mimicked the situation of three non-chiral bosons. A similar thing will happen with the chiral $2k$ -form studied in this paper. The compactification on T^{10} can be viewed as compactification on $T^2 \times T^8$. Effectively, the thirty-five degrees of freedom of the four-form potential (which is the **35_c** representation of the Spin(8) little group for 10d massless states) behave like thirty-five 2d non-chiral scalars.

Our formula has the property that in the case of T^{2p} for odd $p > 1$, there is no spin structure dependence. The situation can be different for compactifications on more general manifolds [5, 6, 9]. A general analysis of the correlation functions of $2k$ -chiral forms, including the $SL(2 + 4k, \mathcal{Z})$ modular invariance of the torus compactification, is provided by Henningson, Nilsson, and Salomonson in [9] using holomorphic factorization.

Section 2 describes the application of the method of [1] to the Ramond-Ramond chiral four-form of Type IIB string theory. Section 3 generalizes our results to spacetime dimension $2 + 4k$. In sections 3 and 4, from our calculation of the partition function, we identify the relevant period matrix, omega function, and the A- and B-lattices, as introduced in [2]. For the case $p = 3$, we show how for a rectangular torus our period matrix corresponds to a direct computation of the periods of the A- and B-cycles, and this can be extended easily for general p .

2. The G_5 Partition Function on T^{10}

We start with a chiral four-form in ten dimensions B_{MNRs} with a self-dual five-form field strength $H_{LMNRS} = \partial_L B_{MNRS} + \partial_M B_{NRS L} + \partial_N B_{RSLM} + \partial_R B_{SLMN} + \partial_S B_{LMNR}$ ($1 \leq L, M, N \leq 10$), which satisfies

$$H_{LMNRS}(\vec{\theta}, \theta^{10}) = \frac{1}{5! \sqrt{-G}} G_{LL'} G_{MM'} G_{NN'} G_{RR'} G_{SS'} \epsilon^{L'M'N'R'S'PQTUV} H_{PQTUV}(\vec{\theta}, \theta^{10}) \quad (2.1)$$

where G_{MN} is the 10d metric. Following [10, 1], we use this equation of motion to eliminate the components $H_{10 mnr s}$ in terms of the other components $H_{lmnr s}$ with $l, m, n, r, s = 1 \dots 9$. The independent fields H_{lmnpq} form a totally antisymmetric tensor which is no longer self-dual. To describe the partition function of the chiral four-form, we use their Hamiltonian and momenta which are written in a 9d covariant way:

$$\mathcal{H} = \frac{1}{2(5!)} \int_0^{2\pi} d\theta^1 \dots d\theta^9 \sqrt{G_9} G_9^{ll'} G_9^{mm'} G_9^{nn'} G_9^{pp'} G_9^{qq'} H_{lmnpq}(\vec{\theta}, \theta^{10}) H_{l'm'n'p'q'}(\vec{\theta}, \theta^{10}) \quad (2.2)$$

$$P_l = -\frac{5}{2(5!)^2} \int_0^{2\pi} d\theta^1 \dots d\theta^9 \epsilon^{rsvwumnpq} H_{umnpq}(\vec{\theta}, \theta^{10}) H_{lrsvw}(\vec{\theta}, \theta^{10}) \quad (2.3)$$

where now indices are raised with the 9d-metric G_9^{mn} and $\epsilon_{123456789} \equiv G_9 \epsilon^{123456789} = G_9$.

An arbitrary flat metric G_{MN} on T^{10} is a function of 55 parameters and can be represented by the line element

$$ds^2 = R_1^2 (d\theta^1 - \alpha d\theta^{10})^2 + R_{10}^2 (d\theta^{10})^2 + \sum_{i,j=2 \dots 9} g_{ij} (d\theta^i - \beta^i d\theta^1 - \gamma^i d\theta^{10}) (d\theta^j - \beta^j d\theta^1 - \gamma^j d\theta^{10}). \quad (2.4)$$

Above, $0 \leq \theta^I \leq 2\pi$, $1 \leq I \leq 10$ and we have singled out directions 1 and 10, in accordance with viewing T^{10} as $T^2 \times T^8$. The direction 10 will be our time

direction. The 55 parameters are R_1, R_{10}, g_{ij} (an 8d metric), β^i, γ^j (the angles between directions 1 and i, j respectively); and α is related to the angle between 1 and 10. As in string theory [8], the partition function is given in terms of the Hamiltonian and momenta by

$$Z(R_1, R_{10}, g_{ij}, \alpha, \beta^i, \gamma^i) = \text{tr} \exp\{-t\mathcal{H} + i2\pi\alpha P_1 + i2\pi(\alpha\beta^i + \gamma^i)P_i\} \quad (2.5)$$

where $t = 2\pi R_{10}$. (Notice the metric in (2.4) is one that has been rotated to Euclidean signature.) The partition function (2.5) is by construction $SL(9, \mathcal{Z})$ invariant, due to the underlying $SO(9)$ invariance in the coordinate space we have labelled $l = 1\dots 9$. We will show it is also $SL(2, \mathcal{Z})$ invariant in the directions 1 and 10. The combination of these two invariances yields the $SL(10, \mathcal{Z})$ invariance of the automorphic form given by Z in (2.20), along the lines of [1].

The Zero Modes of the Partition Function

To trace on the zero mode operators in (2.1), we express the Hamiltonian (2.2) and momenta (2.3) in terms of the metric parameters in (2.4)

$$\begin{aligned} -t\mathcal{H} + i2\pi\alpha P_1 + i2\pi(\alpha\beta^i + \gamma^i)P_i &= \left(-\frac{\pi}{5!}R_{10}R_1\sqrt{g}g^{ii'}g^{jj'}g^{kk'}g^{pp'}g^{qq'}H_{ijkpq}H_{i'j'k'p'q'}\right. \\ &\quad - \frac{\pi}{(4!)^2|\tau|^2}\frac{R_{10}}{R_1}\sqrt{g}4!g^{[jj'}g^{kk'}g^{pp'}g^{qq']}H_{ijkpq}H_{i'j'k'p'q'}\gamma^i\gamma^{i'} \\ &\quad \left.- \pi\mathcal{A}^{jkpqj'k'p'q'}\frac{1}{(4!)^2}[H_{1jkpq} + x_{jkpq}][H_{1j'k'p'q'} + x_{j'k'p'q'}]\right) \end{aligned} \quad (2.6)$$

where $g^{[jj'}g^{kk'}g^{pp'}g^{qq']} \equiv \frac{1}{4!}(g^{jj'}g^{kk'}g^{pp'}g^{qq'} - \dots)$ has $4!$ terms antisymmetric in $jkpq$,

$$x_{jkpq} \equiv \beta^i H_{ijkpq} + \frac{i}{(4!)^2}\gamma^i \mathcal{A}^{-1}_{jkpqj'k'p'q'} \epsilon^{j'k'p'q'ghg'h'} H_{ighg'h'}, \quad (2.7)$$

$$\mathcal{A}^{jkpqj'k'p'q'} \equiv \frac{R_{10}}{R_1}\sqrt{g}4!g^{[jj'}g^{kk'}g^{pp'}g^{qq']} + i\alpha\epsilon^{jkpqj'k'p'q'}, \quad (2.8)$$

and $\tau \equiv \alpha + i\frac{R_{10}}{R_1}$ is the modular parameter of the two-torus.

The trace in the partition function (2.5) is over all independent Fock space operators which appear in the normal mode expansion of the free massless tensor gauge field $B_{MNR S}$. The zero mode eigenvalues of the seventy fields H_{1jkpq} are labeled with the integers n_1, \dots, n_{70} in \mathcal{Z}^{70} , while the zero modes of the fifty six fields H_{ijkpq} are labeled by the integers n_{71}, \dots, n_{126} . Here $2 \leq i, j, k, p, q, \leq 9$. We make this subdivision since the H_{1jkpq} remain non-zero in the “2-d massless” limit obtained when the momenta on the eight-torus are zero, *i.e.* $p_i = 0$. Since only the “2-d massless” oscillator modes contribute to the anomaly, only this subset of seventy zero modes will be needed for the cancellation. In general, on a flat torus of any dimension $2 + 4k$, the zero modes can be separated into $H_{ijkpq\dots}$ and $H_{1jkpq\dots}$. Since the physical degrees of freedom of the chiral gauge field $B_{MNPQ\dots}$ are described by the rank $2k$ complex antisymmetric tensor representation of the little group $SO(4k)$, the number of physical degrees of freedom appearing in the oscillator calculation will always be equal to the number of zero modes in the set $H_{1jkpq\dots}$, so the anomaly will always cancel on the torus.

The contribution from the zero modes is then

$$\begin{aligned}
Z_{\text{zero modes}} = & \sum_{n_{71}, \dots, n_{126} \in \mathcal{Z}^{56}} \exp \left\{ -\pi \frac{R_{10} R_1}{5!} \sqrt{g} g^{ii'} g^{jj'} g^{kk'} g^{pp'} g^{qq'} H_{ijkpq} H_{i'j'k'p'q'} \right. \\
& - \frac{\pi}{(4!)^2 |\tau|^2} \frac{R_{10}}{R_1} \sqrt{g} 4! g^{[jj'} g^{kk'} g^{pp'} g^{qq']} H_{ijkpq} H_{i'j'k'p'q'} \gamma^i \gamma^{i'} \} \\
& \cdot \sum_{n_1, \dots, n_{70} \in \mathcal{Z}^{70}} e^{-\pi(n+x) \cdot A \cdot (n+x)}
\end{aligned} \tag{2.9}$$

where A is a rank 70 matrix with $A_{11} = \mathcal{A}^{23452345}$, $A_{170} = \mathcal{A}^{23456789}, \dots$; $x_1 = x_{2345}, \dots, x_{70} = x_{6789}$; and $H_{12345} = n_1, \dots, H_{16789} = n_{70}$. The description of $Z_{\text{zero modes}}$ by the Riemann theta function $\Theta \left[\begin{smallmatrix} \vec{0} \\ 0 \end{smallmatrix} \right] (\vec{0}, \mathcal{T}_{IJ})$, defined as in [11], with the 126x126 symmetric non-singular complex period matrix \mathcal{T}_{IJ} is discussed in section 3.

To check the modular invariance of the partition function, we argue as follows. In analogy with the modular group $SL(2, \mathcal{Z})$ on the two-torus which can be gen-

erated by the two transformations $\tau \rightarrow -\tau^{-1}$ and $\tau \rightarrow \tau + 1$, the mapping class group of the n -torus, *i.e.* the modular group $SL(n, \mathcal{Z})$, can be also generated by just two transformation as well [12]. The transformation

$$R_1 \rightarrow R_1|\tau|, R_{10} \rightarrow R_{10}|\tau|^{-1}, \alpha \rightarrow -|\tau|^{-2}\alpha, \beta^i \rightarrow \gamma^i, \gamma^i \rightarrow -\beta^i, g_{ij} \rightarrow g_{ij} \quad (2.10)$$

is an $SL(10, \mathcal{Z})$ transformation which leaves invariant the line element (2.4) if $d\theta^1 \rightarrow d\theta^{10}$, $d\theta^{10} \rightarrow -d\theta^1$, $d\theta^i \rightarrow d\theta^i$. It is the generalization of the usual $SL(2, \mathcal{Z})$ modular transformation $\tau \rightarrow -\tau^{-1}$. It can be checked along the lines of [1] that (2.10) differs from a generator of $SL(10, \mathcal{Z})$ only up to an $SL(9, \mathcal{Z})$ transformation. Since we have $SL(9, \mathcal{Z})$ invariance, invariance under (2.10) is therefore sufficient to prove invariance under that $SL(10, \mathcal{Z})$ generator.

In order to study the properties of the partition function under (2.10), it is convenient to use the Poisson summation formula [8]

$$\sum_{n \in \mathcal{Z}^p} e^{-\pi(n+x) \cdot A \cdot (n+x)} = (\det A)^{-\frac{1}{2}} \sum_{n \in \mathcal{Z}^p} e^{-\pi n \cdot A^{-1} \cdot n} e^{2\pi i n \cdot x}. \quad (2.11)$$

to reexpress the sum over the zero modes n_1, \dots, n_{70} . After such resummation, we obtain

$$\begin{aligned} Z = Z_{osc} \cdot & \sum_{n_{71}, \dots, n_{126} \in \mathcal{Z}^{56}} \exp\left\{-\pi \frac{R_{10} R_1}{5!} \sqrt{g} g^{ii'} g^{jj'} g^{kk'} g^{pp'} g^{qq'} H_{ijkpq} H_{i'j'k'p'q'} \right. \\ & \left. - \frac{\pi}{(4!)^2 |\tau|^2} \frac{R_{10}}{R_1} \sqrt{g} 4! g^{[jj'} g^{kk'} g^{pp'} g^{qq']} H_{ijkpq} H_{i'j'k'p'q'} \gamma^i \gamma^{i'} \right\} \\ & \cdot \sum_{n_1, \dots, n_{70} \in \mathcal{Z}^{70}} \exp\left\{-\frac{\pi}{(4!)^2 |\tau|^2} \left(\frac{R_{10}}{R_1} \sqrt{g} 4! g^{[jj'} g^{kk'} g^{pp'} g^{qq']} - i \frac{\alpha}{g} \epsilon_{jkpqj'k'p'q'} \right) H_1^{jkpq} H_1^{j'k'p'q'} \right. \\ & \left. + \frac{2\pi}{4!} (i H_1^{jkpq} (\beta^i H_{ijkpq} + \mathcal{A}^{-1}_{jkj'k'} \epsilon^{j'k'gh} H_{igh} \gamma^i \frac{i}{(4!)^2}) \right\} \cdot \frac{1}{\sqrt{\det A}} \end{aligned} \quad (2.12)$$

where the H_1^{jkpq} are defined to be the integers H_{1jkpq} , and \mathcal{A}^{-1} is given by

$$\mathcal{A}^{-1}_{jkpqj'k'p'q'} = |\tau|^{-2} \left\{ \frac{R_{10}}{R_1} \frac{1}{\sqrt{g}} 4! g^{[jj'} g^{kk'} g^{pp'} g^{qq']} - i \frac{\alpha}{g} \epsilon_{jkpqj'k'p'q'} \right\}, \quad (2.13)$$

and A defined below (2.9) can be proved to have $\det A = |\tau|^{70}$. With this rewrit-

ing (2.12) , it is more convenient to show that under (2.10) $Z_{\text{zero modes}}$ changes according to

$$Z_{\text{zero modes}}(R_1|\tau|, R_{10}|\tau|^{-1}, g_{ij}, -\alpha|\tau|^2, \gamma^i, -\beta^i) = (\det A)^{\frac{1}{2}} Z_{\text{zero modes}}(R_1, R_{10}, g_{ij}, \alpha, \beta^i, \gamma^i). \quad (2.14)$$

Therefore (2.10) generates an anomaly that needs to be cancelled by the sum over the oscillators. (The determinant $\det A = |\tau|^{70}$ does not depend on the other parameters of the torus metric since A and A^{-1} are effectively proportional to each other, such that $1 = \det A \det A^{-1} = (\det A)^2 |\tau|^{-140}$.)

The other $SL(10, \mathcal{Z})$ generator, the analogue of $\tau \rightarrow \tau + 1$, instead leaves invariant both the zero modes and the oscillators as in [1], and we will not discuss it further.

The Anomaly Cancellation

To sum over the oscillators, the key point to realize is that the chiral four-form in ten dimensions has 35 independent degrees of freedom. Hence the partition function can be written as

$$Z = Z_{\text{zero modes}} \cdot \text{tr} e^{-2i\pi \sum_{\vec{p} \neq 0} p_{10} \mathcal{B}_{\vec{p}}^{\kappa\dagger} B_{\vec{p}}^{\kappa} - \pi R_{10} \sum_{\vec{p}} \sqrt{G_9^{mn} p_m p_n} \delta^{\kappa\kappa}} \quad (2.15)$$

where $1 \leq \kappa \leq 35$, and $\mathcal{B}_{\vec{p}}^{\kappa\dagger}, B_{\vec{p}}^{\kappa}$ are the creation and annihilation operators associated with each degree of freedom, obeying the canonical commutation relations

$$[\mathcal{B}_{\vec{p}}^{\kappa\dagger}, B_{\vec{p}}^{\lambda}] = \delta^{\kappa\lambda} \delta_{\vec{p}, \vec{p}^*}. \quad (2.16)$$

The momenta $p_l = n_l \in \mathcal{Z}^9$ are integers since we are compactifying to the torus, and the momentum p_{10} can be written explicitly in terms of the other components by using the equation of motion as in [1]

$$\begin{aligned} p_{10} &= -\frac{G^{10m}}{G^{1010}} p_m - i \sqrt{\frac{G_9^{mn}}{G^{1010}}} p_m p_n \\ &= -\alpha p_1 - (\alpha \beta^i + \gamma^i) p_i - i R_{10} \sqrt{G_9^{mn} p_m p_n} \end{aligned} \quad (2.17)$$

where $2 \leq i \leq 9; 1 \leq m, n \leq 9$.

The standard Fock space computation $\text{tr} \omega^{\sum_p p a_p^\dagger a_p} = \prod_p \sum_{k=0}^{\infty} \langle k | \omega^{p a_p^\dagger a_p} | k \rangle = \prod_p \frac{1}{1 - \omega^p}$ is used to do the trace on the oscillators in (2.15) giving

$$Z = Z_{\text{zero modes}} \cdot \left(e^{-\pi R_{10} \sum_{\vec{n}} \sqrt{G_9^{lm} n_l n_m}} \prod_{\vec{n} \neq \vec{0}} \frac{1}{1 - e^{-i2\pi p_{10}}} \right)^{35}. \quad (2.18)$$

(2.18) is manifestly $SL(9, \mathcal{Z})$ invariant since p_{10} is. The vacuum energy $\sum_{\vec{n}} \sqrt{G_9^{lm} n_l n_m}$ is a divergent sum which we regularize in a way that preserves the $SL(9, \mathcal{Z})$ invariance, similar to the calculation performed in [1].

The regularized version is then

$$Z = Z_{\text{zero modes}} \cdot \left(e^{-R_{10} \pi^{-5} 12 \sum_{\vec{n} \neq \vec{0}} \frac{\sqrt{G_9}}{(G_{lm} n_l n_m)^5}} \prod_{\vec{n} \neq \vec{0}} \frac{1}{1 - e^{-2\pi R_{10} \sqrt{G_9^{lm} n_l n_m} + i2\pi \alpha n_1 + i2\pi (\alpha \beta^i + \gamma^i) n_i}} \right)^{35}. \quad (2.19)$$

$Z_{\text{zero modes}}$ is given in (2.9). We separate the product on $\vec{n} = (n, n_\perp) \neq \vec{0}$ into a product on (all $n, n_\perp \neq 0$) and on ($n \neq 0, n_\perp = 0$), where $n_\perp = n_i$ is the momentum on the eight torus. Then (2.19) becomes

$$\begin{aligned} Z &= Z_{\text{zero modes}} \cdot \left(e^{\frac{R_{10}}{\pi R_1} \zeta(2)} \prod_{n_1 \neq 0} \frac{1}{1 - e^{2\pi i (\alpha n_1 + i \frac{R_{10}}{R_1} |n_1|)}} \right)^{35} \\ &\cdot \left(\prod_{n_i \neq (0,0,0,0,0,0,0,0)} e^{-2\pi R_{10} \langle H \rangle_{n_\perp}} \prod_{n_1 \in \mathcal{Z}} \frac{1}{1 - e^{-2\pi R_{10} \sqrt{G_9^{lm} n_l n_m} + i2\pi \alpha n_1 + i2\pi (\alpha \beta^i + \gamma^i) n_i}} \right)^{35} \\ &= Z_{\text{zero modes}} \cdot \left(\eta(\tau) \bar{\eta}(\bar{\tau}) \right)^{-35} \\ &\cdot \left(\prod_{n_i \neq (0,0,0,0,0,0,0,0)} e^{-2\pi R_{10} \langle H \rangle_{n_\perp}} \prod_{n_1 \in \mathcal{Z}} \frac{1}{1 - e^{-2\pi R_{10} \sqrt{G_9^{lm} n_l n_m} + i2\pi \alpha n_1 + i2\pi (\alpha \beta^i + \gamma^i) n_i}} \right)^{35} \end{aligned} \quad (2.20)$$

where $\tau \equiv \alpha + i \frac{R_{10}}{R_1}$ and

$$\langle H \rangle_{n_\perp} = -|n_\perp|^2 R_1 \sum_{n^1=1}^{\infty} \cos(n_\perp \cdot \beta 2\pi n^1) [K_2(2\pi n^1 R_1 |n_\perp|) - K_0(2\pi n^1 R_1 |n_\perp|)] \quad (2.21)$$

as given in appendix A of [1], but with n_\perp now given by the eight-vector n_i . In (2.20) we have separated the contribution of the ‘2d massless’ scalars (with zero

momentum $n_\perp = 0$) from the contribution of the ‘2d massive’ scalars. The latter modes are associated with the eight-torus momentum $n_\perp \neq 0$ and correspond to massive bosons on the 2-torus. Their partition function at fixed n_\perp is

$$e^{-2\pi R_{10} \langle H \rangle_{n_\perp}} \prod_{n_1 \in \mathcal{Z}} \frac{1}{1 - e^{-2\pi R_{10} \sqrt{G_9^{lm} n_l n_m} + i2\pi \alpha n_1 + i2\pi (\alpha \beta^i + \gamma^i) n_i}} \quad (2.22)$$

and is $SL(2, \mathcal{Z})$ symmetric by itself, since there is no anomaly for massive states. The only piece of (2.20) that has an $SL(2, \mathcal{Z})$ anomaly is the one associated with the ‘2d massless’ modes

$$e^{\frac{R_{10}}{\pi R_1} \zeta(2)} \prod_{n_1 \neq 0} \frac{1}{1 - e^{2\pi i (\alpha n_1 + i \frac{R_{10}}{R_1} |n_1|)}} = (\eta(\tau) \bar{\eta}(\bar{\tau}))^{-1}$$

where the Dedekind eta function $\eta(\tau) \equiv e^{\frac{\pi i \tau}{12}} \prod_{n=1}^{\infty} (1 - e^{2\pi i \tau n})$, and $\zeta(2) = \frac{\pi^2}{6}$ comes from regularizing the divergent sum $\sum_n |n|$. Under the $SL(2, \mathcal{Z})$ transformation (2.10) of sect. 3, $\tau \rightarrow -\frac{1}{\tau}$ and

$$(\eta(\tau) \bar{\eta}(\bar{\tau}))^{-35} \rightarrow |\tau|^{-35} (\eta(\tau) \bar{\eta}(\bar{\tau}))^{-35}. \quad (2.23)$$

This is how the oscillator anomaly cancels the zero mode anomaly in (2.14). Hence the combination $Z_{\text{zero modes}} \cdot (\eta(\tau) \bar{\eta}(\bar{\tau}))^{-35}$ is $SL(2, \mathcal{Z})$ invariant, and the expression for the partition function Z derived in (2.20) is $SL(10, \mathcal{Z})$ invariant.

3. $SL(2p, \mathcal{Z})$ Automorphic Form

Following the generalization of the last section, we can now give the partition function for the self-dual p -form on T^{2p} for odd $p > 1$. It is invariant under $SL(2p, \mathcal{Z})$. The number of physical degrees of freedom F of the chiral $(p-1)$ -form abelian gauge field with self-dual field strength $H_{i_1 \dots i_p}$ compactified on T^{2p} for odd $p > 1$ is $F = \frac{1}{2} \frac{(2p-2)!}{((p-1)!)^2}$. The total number of zero modes is $P = \frac{1}{2} \frac{(2p)!}{(p!)^2}$, which is half the middle Betti number on the torus, $b_p(T^{2p}) = \dim H^p(T^{2p}; \mathcal{Z}) = \binom{2p}{p} = 2P$. Viewing T^{2p} as $T^2 \times T^{4k}$, we see the oscillator trace again breaks up into a product over ‘2d-massless scalars’ given by $(\eta(\tau)\bar{\eta}(\bar{\tau}))^{-F}$ and ‘2d-massive’ scalars so that

$$\begin{aligned} Z(R_1, R_{2p}, g_{ij}, \alpha, \beta^i, \gamma^i) &= \text{tr} \exp\{-t\mathcal{H} + i2\pi\alpha P_1 + i2\pi(\alpha\beta^i + \gamma^i)P_i\} \\ &= Z_{\text{zero modes}} \cdot (\eta(\tau)\bar{\eta}(\bar{\tau}))^{-F} \\ &\quad \cdot \left(\prod_{n_i \neq (0, \dots, 0)} e^{-2\pi R_{2p} \langle H \rangle_{n_\perp}} \prod_{n_1 \in \mathcal{Z}} \frac{1}{1 - e^{-2\pi R_{2p} \sqrt{G_{2p-1}^{lm}} n_l n_m + i2\pi\alpha n_1 + i2\pi(\alpha\beta^i + \gamma^i)n_i}} \right)^F \end{aligned} \quad (3.1)$$

where

$$\begin{aligned} Z_{\text{zero modes}} &= \sum_{n_{2F+1}, \dots, n_P \in \mathcal{Z}^{P-2F}} \exp\left\{-\pi \frac{R_{2p} R_1}{p!} \sqrt{g} g^{i_1 i'_1} \dots g^{i_p i'_p} H_{i_1 \dots i_p} H_{i'_1 \dots i'_p} \right. \\ &\quad \left. - \frac{\pi}{(p-1)! |\tau|^2} \frac{R_{2p}}{R_1} \sqrt{g} g^{[j_1 j'_1} \dots g^{j_{p-1} j'_{p-1}]} H_{i j_1 \dots j_{p-1}} H_{i' j'_1 \dots j'_{p-1}} \gamma^i \gamma^{i'} \right\} \\ &\quad \cdot \sum_{n_1, \dots, n_{2F} \in \mathcal{Z}^{2F}} e^{-\pi(n+x) \cdot A \cdot (n+x)} \end{aligned} \quad (3.2)$$

with

$$\begin{aligned} \mathcal{A}^{j_1 \dots j_{p-1} j'_1 \dots j'_{p-1}} &= \frac{R_{2p}}{R_1} \sqrt{g} (p-1)! g^{[j_1 j'_1} \dots g^{j_{p-1} j'_{p-1}]} + i\alpha \epsilon^{j_1 \dots j_{p-1} j'_1 \dots j'_{p-1}}, \\ x_{j_1 \dots j_{p-1}} &\equiv \beta^i H_{i j_1 \dots j_{p-1}} + \frac{i}{((p-1)!)^2} \gamma^i \mathcal{A}^{-1}_{j_1 \dots j_{p-1} j'_1 \dots j'_{p-1}} \epsilon^{j'_1 \dots j'_{p-1} k_1 \dots k_{p-1}} H_{i k_1 \dots k_{p-1}}, \\ \mathcal{A}^{-1}_{j_1 \dots j_{p-1} j'_1 \dots j'_{p-1}} &= |\tau|^{-2} \left\{ \frac{R_{2p}}{R_1} \frac{1}{\sqrt{g}} (p-1)! g_{[j_1 j'_1} \dots g_{j_{p-1} j'_{p-1}]} - i \frac{\alpha}{g} \epsilon_{j_1 \dots j_{p-1} j'_1 \dots j'_{p-1}} \right\} \end{aligned}$$

and the transverse directions corresponding to directions on T^{4k} are $i_1, j_1, \dots = 2, \dots, 2p-1$, for $p = 1 + 2k$.

Our proof shows the invariance of (3.1) under the generators of $SL(2p, \mathcal{Z})$ which are given by the two transformations generalizing $\tau \rightarrow \tau - 1$ and $\tau \rightarrow -\frac{1}{\tau}$:

$$\begin{aligned} R_1 &\rightarrow R_1, R_{2p} \rightarrow R_{2p}, \alpha \rightarrow \alpha - 1, \beta^i \rightarrow \beta^i, \gamma^i \rightarrow \gamma^i + \beta^i, g_{ij} \rightarrow g_{ij}; \\ R_1 &\rightarrow R_1|\tau|, R_{2p} \rightarrow R_{2p}|\tau|^{-1}, \alpha \rightarrow -|\tau|^{-2}\alpha, \beta^i \rightarrow \gamma^i, \gamma^i \rightarrow -\beta^i, g_{ij} \rightarrow g_{ij}. \end{aligned} \quad (3.3)$$

We can express (3.1) as $Z(R_1, R_{2p}, g_{ij}, \alpha, \beta^i, \gamma^i) = \frac{\Theta}{\Delta}$ where our explicit calculation of Δ gives

$$\begin{aligned} \Delta &= (\eta(z)\bar{\eta}(\bar{z}))^{\frac{1}{2} \frac{(2p-2)!}{((p-1)!)^2}} \\ &\cdot \left(\prod_{n_i \neq (0, \dots, 0)} e^{-2\pi R_{2p} \langle H \rangle_{n_\perp}} \prod_{n_1 \in \mathcal{Z}} \frac{1}{1 - e^{-2\pi R_{2p} \sqrt{G_{2p-1}^{lm} n_i n_m + i2\pi \alpha n_1 + i2\pi (\alpha \beta^i + \gamma^i) n_i}}} \right)^{-\frac{1}{2} \frac{(2p-2)!}{((p-1)!)^2}}. \end{aligned} \quad (3.4)$$

Δ is the trace over the non-zero modes of the self-dual G_p on the manifold T^{2p} , for odd $p > 1$. For a different manifold M^{2p} , the function Δ would be different.

The zero mode contribution $Z_{\text{zero modes}}$ is a Riemann theta function (with zero-valued characteristics) $\Theta \begin{bmatrix} \vec{0} \\ \vec{0} \end{bmatrix} (\vec{0}, \mathcal{T}) = \sum_{n_1, \dots, n_P \in \mathcal{Z}^P} \exp\{i\pi n^I n^J \mathcal{T}_{IJ}\}$ where the rank P complex symmetric period matrix \mathcal{T}_{IJ} can be reconstructed from (3.2). This particular theta function is one of the 2^{2P} spin structures on a Riemann surface of genus P . (A Riemann surface Σ_g of genus g has 2^{2g} spin structures which transform into each other under $Sp(2g, \mathcal{Z})$, the mapping class group of Σ_g). According to our proof, this particular theta function $\Theta \begin{bmatrix} \vec{0} \\ \vec{0} \end{bmatrix} (\vec{0}, \mathcal{T})$ multiplied by $(\eta(\tau)\bar{\eta}(\bar{\tau}))^{-F}$ is actually invariant under an $SL(2p, \mathcal{Z})$ subgroup of $Sp(2P, \mathcal{Z})$. (On T^n , there are 2^n spin structures, which in general transform into each other under $SL(n, \mathcal{Z})$).

4. Calculation of the Period Matrix from A- and B-cycles

We now discuss our result in terms of the description via the period lattice and omega function given in [2,3] for calculating partition functions on arbitrary spacetimes of dimension $2 + 4k$. We identify the period matrix in our expression, and show how for the simple case of a rectangular torus, it indeed could have been constructed directly from the A- and B-cycle periods. For the general flat metric, however, our formula provides a simple expression for the B-lattice basis vectors (on the torus), which otherwise would be more complicated to compute directly.

To this end, we focus on the theta function given by our formula in the simplest case of the fivebrane chiral two-form ($p = 3$). In general, our p -forms are closed, since they are locally exact $G_p = dB$, and their periods $\frac{1}{2\pi} \int_{U_p} G_p$, defined with respect to p -cycles U_p will take values in the period lattice $\Lambda = H^p(T^{2p}; \mathbf{Z}) = \mathbf{Z}^{b_p(T^{2p})} = \mathbf{Z}^{\binom{2p}{p}} = \mathbf{Z}^{2P}$, where $2P \equiv \binom{2p}{p}$. For the fivebrane on T^6 , the period lattice is $\Lambda = H^3(T^6; \mathbf{Z}) = \mathbf{Z}^{20}$. The lattice $\Lambda = \mathbf{Z}^{20}$ has 2^{20} theta functions and the number of zero modes is $P = 10$. The 2^{20} theta functions correspond to those defined on the Riemann surface Σ_{10} of genus $g = 10$. $Z_{\text{zero modes}}$ is given by the one that has zero characteristics and is invariant (up to a c-number multiple) under $SL(6, \mathcal{Z})$. To display its 10×10 period matrix \mathcal{T}_{IJ} we find, from rewriting [1] the explicit sum over the zero modes in (3.1), that the relevant theta function is

$$\begin{aligned}
Z_{\text{zero modes}} &= \sum_{n_1, \dots, n_{10} \in \mathcal{Z}^{10}} \exp \left\{ -\pi \frac{R_6 R_1}{6} \sqrt{g} g^{ii'} g^{jj'} g^{kk'} H_{ijk} H_{i'j'k'} \right. \\
&\quad - \frac{\pi}{4} \left[\frac{R_6}{R_1} \sqrt{g} (g^{jj'} g^{kk'} - g^{jk'} g^{kj'}) + i\alpha \epsilon^{jkj'k'} \right] H_{1jk} H_{1j'k'} \\
&\quad - \frac{\pi}{2} \left[\frac{R_6}{R_1} \sqrt{g} (g^{jj'} g^{kk'} - g^{jk'} g^{kj'}) + i\alpha \epsilon^{jkj'k'} \right] \beta^i H_{1jk} H_{ij'k'} \\
&\quad - \frac{\pi}{4} \left[\frac{R_6}{R_1} \sqrt{g} (g^{jj'} g^{kk'} - g^{jk'} g^{kj'}) + i\alpha \epsilon^{jkj'k'} \right] \beta^i \beta^{i'} H_{ijk} H_{i'j'k'} \\
&\quad \left. - \frac{i\pi}{2} \gamma^i \epsilon^{jkj'k'} H_{1jk} H_{ij'k'} \right\} \\
&= \sum_{n_1, \dots, n_{10} \in \mathcal{Z}^{10}} \exp \{ i\pi n^I n^J \mathcal{T}_{IJ} \} = \sum_{x \in \Lambda_1} \Omega(x) e^{i\pi(x, \mathcal{T}x)} = \Theta(\vec{0}, \mathcal{T}_{IJ})
\end{aligned} \tag{4.1}$$

where $H_{123} = n_1$, $H_{124} = n_2$, $H_{125} = n_3$, $H_{134} = n_4$, $H_{135} = n_5$, $H_{145} = n_6$, $H_{234} = n_7$, $H_{235} = n_8$, $H_{245} = n_9$, $H_{345} = n_{10}$. So we identify $\Lambda_1 = \mathbf{Z}^{10}$, the omega function restricted to Λ_1 as $\Omega(x) = 1$, and the 10×10 period matrix \mathcal{T}_{IJ} in terms of the parameters of the metric, *i.e.* $\mathcal{T}_{11} = i \frac{R_6}{R_1} \sqrt{g}(g^{22}g^{33} - g^{23}g^{32})$, etc. For an arbitrary flat metric on T^6 , all components of \mathcal{T}_{IJ} are non-vanishing.

For simplicity, we reduce to the case of a metric for a rectangular torus. Then $G_{11} = R_1^2$, $g_{22} = R_2^2$, $g_{33} = R_3^2$, $g_{44} = R_4^2$, $g_{55} = R_5^2$, $G_{66} = R_6^2$ (and all other parameters $g_{ij} = 0$ for $i \neq j$, and $\alpha, \beta^i, \gamma^i = 0$.) The period matrix which occurs in (4.1) then has only non-vanishing diagonal components given by

$$\begin{aligned} \mathcal{T}_{11} &= i \frac{R_6 R_5 R_4}{R_1 R_2 R_3}, & \mathcal{T}_{22} &= i \frac{R_6 R_3 R_5}{R_1 R_2 R_4}, \\ \mathcal{T}_{33} &= i \frac{R_6 R_3 R_4}{R_1 R_2 R_5}, & \mathcal{T}_{44} &= i \frac{R_6 R_2 R_5}{R_1 R_3 R_4}, \\ \mathcal{T}_{55} &= i \frac{R_6 R_2 R_4}{R_1 R_3 R_5}, & \mathcal{T}_{66} &= i \frac{R_6 R_2 R_3}{R_1 R_4 R_5}, \\ \mathcal{T}_{77} &= i \frac{R_1 R_6 R_5}{R_2 R_3 R_4}, & \mathcal{T}_{88} &= i \frac{R_1 R_6 R_4}{R_2 R_3 R_5}, \\ \mathcal{T}_{99} &= i \frac{R_1 R_6 R_3}{R_2 R_4 R_5}, & \mathcal{T}_{10\,10} &= i \frac{R_1 R_6 R_2}{R_3 R_4 R_5}. \end{aligned} \tag{4.2}$$

Alternatively, we can derive these components of the period matrix directly by picking the ‘B-periods’ as follows. Using the method of [2], we construct the period matrix of the theta function by noting that for self-dual quantum fields we should sum over only half the periods, *i.e.* those which lie on the A-lattice. In this case, from [2] we find the lattice of A periods is $\Lambda_1 = H^3(T^5; \mathbf{Z}) = \mathbf{Z}^{10}$, and the lattice of B periods is $\Lambda_2 = H^2(T^5; \mathbf{Z}) = \mathbf{Z}^{10}$, where $\Lambda = \Lambda_1 \oplus \Lambda_2$. But the values the B periods take on reflect the choice of basis vectors of the B-lattice, *i.e.* the periods of the B-cycles lie on the lattice $\Lambda_B = \sum_{I=1}^{10} n_I \mathcal{T}_{IJ}$ where $n_I \in \mathbf{Z}^{10}$. That is to say, the values of the periods $\int_{U_3} H_{LMN} dx^L \wedge dx^M \wedge dx^N$ lie in Λ_B , when U_3 is a B-cycle, for eg. one of ten possible basic cycles corresponding to H_{6MN} .

The A-cycles (whose periods lie on the integer lattice $\Lambda_1 = \sum_{I=1}^{10} n_I \delta_{IJ}$) correspond to the other ten components of H_{LMN} , *i.e.* $H_{321}, H_{421}, \dots, H_{543}$.

From the self-dual equation of motion [1], we have for instance $H_{654} = \frac{1}{\sqrt{-G}} G_{66} G_{55} G_{44} \epsilon^{654321} H_{321}$, i.e.

$$H_{654} = i \frac{R_6 R_5 R_4}{R_1 R_2 R_3} H_{321} . \quad (4.3)$$

Therefore, if the period over the A-cycle corresponding to H_{321} is n , then the period of the B-cycle corresponding to H_{654} is $\mathcal{T}_{11} n$ where $\mathcal{T}_{11} = i \frac{R_6 R_5 R_4}{R_1 R_2 R_3}$. This last sentence is due to the self-dual equation for H_{654} given in (4.3) above. So this gives us a direct computation of \mathcal{T}_{11} , and it matches what we found from our calculation of the partition function (4.2) . (All components \mathcal{T}_{IJ} follow similarly).

5. Conclusions

In this paper we computed explicitly the partition function $Z = \frac{\Theta}{\Delta}$ of the type IIB superstring chiral four-form on a ten-torus, and extended the result for the general chiral $(p-1)$ -form on T^{2p} for all odd $p > 1$. We found expressions for the determinant of the non-zero modes Δ , and the period matrix \mathcal{T}_{IJ} occurring in the theta function describing the zero modes, in terms of the metric parameters of the spacetime torus. We also showed how this period matrix comes in from a direct calculation via the cycles.

In analogy with the $SL(2, \mathcal{Z})$ modular invariance of 2d string theory, the partition function of the self-dual p -form field strength has $SL(2p, \mathcal{Z})$ modular invariance on T^{2p} for odd $p > 1$. Invariance under the mapping class group of the spacetime would not necessarily be the case for more general manifolds. In general, a knowledge of the partition functions for self-dual p-forms on various manifolds may lead to a better formulation of these non-Lagrangian fields and ultimately the description of unitarity in M-theory.

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